Consistency index less affected by the size of pairwise comparison matrix in AHP

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Abstract

In analytic hierarchy process proposed by Saaty, pairwise comparison matrix plays important roles. Priority weights for objects under consideration and index of consistency of judgments are obtained from pairwise comparison matrix. Many indices of consistency have been discussed in the literature. This article examines distributions of two indices for pairwise comparison matrices generated at random. Then, by dividing an existing index by a quadratic, new index of consistency is proposed.

Keywords: Analytic hierarchy process; distribution of index of consistency; random pairwise comparison matrix.

1. Introduction

Analytic hierarchy process (AHP) proposed by Saaty (1980) is a decision making method with subjective judgments. AHP has a characteristic that the structure of decision making is expressed by a hierarchical chart. Another feature of AHP is the mechanism of evaluating priority weights of objects (criteria or alternatives) with the pairwise comparison matrices.

A pairwise comparison matrix is the matrix whose elements express the ratio of priorities of two objects. In order to assess the priority weights of objects from a pairwise comparison matrix, eigenvector method is used popularly. Eigenvector method adopts the principal eigenvector of the matrix as the weights vector of objects. Simultaneously, an index of consistency, Saaty's CI (Saaty, 1980), is obtained from the principal eigenvalue of the matrix. Saaty suggested that the pairwise comparison matrix is sufficiently consistent if $\mathrm{CI} \leq 0.1$. After that, other indices of consistency are

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proposed in the literatures (Brunelli et al., 2013, Crawford and Williams, 1985, Obata et al., 1999, Peláez and Lamata, 2003, Shiraishi and Obata, 2002, Shiraishi et al., 1998).

The degree of consistency varies according to the size of matrix. The levels of consistency of two matrices with same size can be compared by the values of certain index of consistency. However, where two matrices of different size have the same value of certain index, it is questionable whether they have the same level of consistency. The aims of this research are to explore for an index of consistency that is less affected by the size of pairwise comparison matrix and to obtain the reference value of the index that substitute for $\mathrm{CI} = 0.1$.

We recall two indices of consistency, Saaty's CI and $-c_3$ of Shiraishi et al. (Obata et al., 1999, Shiraishi and Obata, 2002, Shiraishi et al., 1998) in Section 2. In Section 3, we introduce three methods for generating random pairwise comparison matrices and examine distributions of the values of indices. As the result, we propose a new index of consistency that is less affected by the size of matrix. In Section 4, we propose the reference value for consistency. We conclude this paper in Section 5.

2. Pairwise comparison matrix and indices of consistency

A pairwise comparison matrix $A = (a_{ij})$ is constructed through the following process. Decision maker compares each pair (i, j), $i, j = 1, \ldots, n$, of n objects, assesses the ratio of priority of these objects as one of the numbers 1, 2, …, 9 and their reciprocals $1, 1/2, \ldots, 1/9$, and sets the element a_{ij} of A to assessed number. Larger number means that object i has higher priority than object j. Since the symmetrical element a_{ji} means relative priority of object j for object i, its value equals reciprocal of a_{ij} inevitably. The diagonal elements mean relative priorities for the same objects, therefore the values are one. Hence, pairwise comparison matrix A has the following form:

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix}$$

The principal eigenvector of such matrix is normalized so that the sum of the elements is one and is regarded as the priority weights of objects.

Let the *true* priority weights of objects be w_i, $i = 1,\ldots,n$, where $\sum_{i=1}^{n} w_i = 1$. If decision maker made completely consistent judgment, the elements of the pairwise comparison matrix satisfied

$$a_{ij} = \frac{w_i}{w_j},\ i, j = 1,\ldots,n.$$

Under this situation, the principal eigenvalue $\lambda_{\max}$ and eigenvector of the matrix are equal to the size of the matrix n and $(w_1,\ldots,w_n)^{\mathrm{T}}$ respectively. However, it is difficult that human judgments become completely consistent. Therefore $\lambda_{\max}$ is not always equal to n. For that reason, it is proposed to consider a degree of deviation of $\lambda_{\max}$ from n as an index which represents consistency of judgment. Saaty proposed

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1}$$

as an index of consistency. He also suggested the pairwise comparison matrix is sufficiently consistent if $\mathrm{CI} \le 0.1$.

After this Saaty's first proposal, many indices of consistency[1] by other approach have been proposed (Brunelli et al., 2013, Crawford and Williams, 1985, Obata et al., 1999, Peláez and Lamata, 2003, Shiraishi and Obata, 2002, Shiraishi et al., 1998).

Shiraishi et al. (Obata et al., 1999, Shiraishi and Obata, 2002, Shiraishi et al., 1998) showed that the coefficient c_3 of degree $n-3$ of the characteristic polynomial of the pairwise comparison matrix always is nonpositive and $c_3 = 0$ is equivalent to $\lambda_{\max} = n$, and proposed to use $-c_3$ as an index of consistency. The coefficient c_3 can be described as

[1] Although it may be denoted by 'index of *in*consistency' on other articles, we use the term 'index of consistency' here. Anyway, smaller value of index indicates higher consistency.

$$c_3 = \sum_{i<j<k} \left(2 - \left(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) \right).$$

Brunelli et al. (2013) investigated agreement between CI and 10 indices of consistency include $-c_3$ when $n = 4,6,8$ by numerical experiments. As the result, it is shown that $-c_3$ has good agreement with CI. Besides, Shiraishi (2015) proofed that $-c_3$ has one-to-one correspondence with CI when $n = 3$ and obtained the expression

$$-c_3 = 8\mathrm{CI}^3 + 24\mathrm{CI}^2 + 18\mathrm{CI}. \quad (1)$$

Most indices, include CI and $-c_3$, tend to become large as the size of matrix n becomes large. Though Saaty proposed $\mathrm{CI} \le 0.1$ as the reference value of good consistency, there is not strong authority. It is questionable that $\mathrm{CI} = 0.1$ with $n = 3$ and $\mathrm{CI} = 0.1$ with $n = 10$ represent the same degree of consistency. Hence, an index of consistency that is less affected by the size of matrix is desired.

3. Consistency index less affected by the size of matrix

In order to examine the variation of the index values, we see the distributions of the index values of pairwise comparison matrices generated at random. We consider three methods for generating random matrices.

Random pairwise comparison matrices

1. Set the diagonal elements of matrix A of size n to one.
2. Sample $n(n-1)/2$ values from $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\}$ at random, and set the upper right elements of A to these values.
3. Set the lower left elements of A to reciprocal of their symmetrical elements.

Highly consistent pairwise comparison matrices

1. Generate n random numbers $w_1, \ldots, w_n$ from [0,1] uniform distribution $\mathrm{U}(0,1)$.
2. Round w_i / w_j to one of $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\}$ [2] and set the (i, j)

[2] The boundary value of rounding is set to the geometric mean of two values, i.e., value between k and $k+1$ is rounded to k if it is smaller than $\sqrt{k(k+1)}$ or to $k+1$ otherwise; value between $1/(k+1)$ and $1/k$ is rounded to $1/(k+1)$ if it is smaller than $1/\sqrt{k(k+1)}$ or to $1/k$ otherwise. Value greater than 9 is rounded to 9 and value smaller than 1/9 is rounded to 1/9.

element of matrix A to the rounded value.

Moderately consistent pairwise comparison matrices

1. Generate n random numbers $w_1,\ldots,w_n$ from [0,1] uniform distribution $\mathrm{U}(0,1)$.
2. Perturb w_i / w_j by multiplying a random number following logarithmic normal distribution $\mathrm{LN}(0,\sigma^2)$ whose mean is one[3].
3. Round the value obtained in Step 2 to one of $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\}$ and set the (i, j) element of matrix A to the rounded value.

The sets of $N = 10000$ matrices generated by each methods are denoted S_1, S_2 and S_3 respectively. We consider $n = 3,4,\ldots,10$ from now.

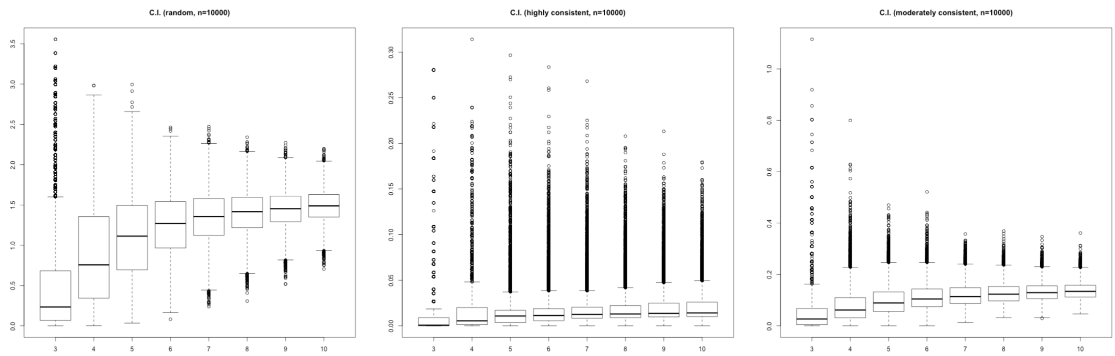

Figure 1: Box plots of CIs for S_1, S_2 and S_3.

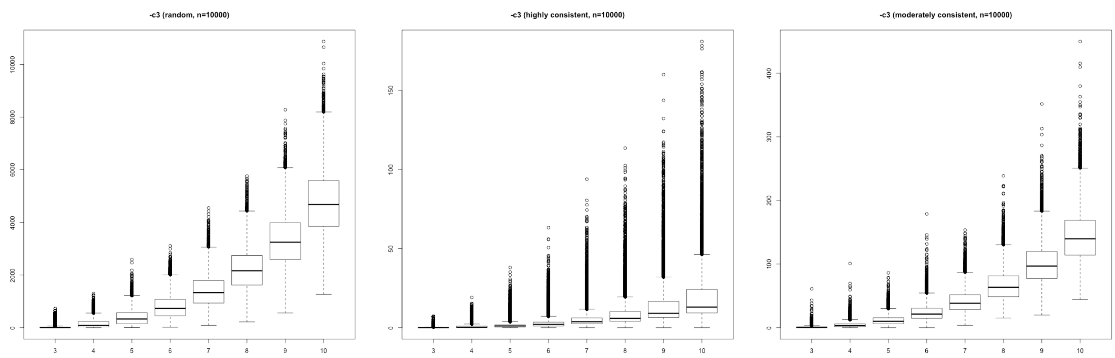

Figure 2: Box plots of $-c_3$ s for S_1, S_2 and S_3.

[3] Parameter σ is arranged to $\ln(1.75)$ in order that CIs of generated matrices are distributed around 0.1.

Fig. 1 shows the distributions of CIs for S_1, S_2 and S_3. Note that matrix whose CI is smaller than 0.1 does not exist in S_1 when $n \geq 6$; CIs of matrices in S_2 are absolutely close to zero; and CIs of matrices in S_3 are distributed around 0.1.

The distributions of $-c_3$ s for S_1, S_2 and S_3 are shown in Fig. 2. About each of the sets, it is supposed that central position of the distribution increase polynomially as the size of matrix n increase.

Then, we performed quadratic polynomial regression[4] that predicts values of $-c_3$ (y) by n:

$$y \sim a + bn + cn^2 .$$

As the results, we found the following regression polynomials:

$$S_1 : 6.207 - 3.359n + 0.4812n^2 = 0.4812(n^2 - 6.981n + 12.90),$$

$$S_2 : 1436 - 770.7n + 109.8n^2 = 109.8(n^2 - 7.018n + 13.08),$$

$$S_3 : 43.23 - 23.27n + 3.319n^2 = 3.319(n^2 - 7.012n + 13.02).$$

All of them roughly have the form of constant multiplication of a quadratic[5]

$$n^2 - 7n + 13. \tag{2}$$

Thus, it is suggested that this quadratic (2) can explain increasing behavior of $-c_3$. Then we expect that an index that is less affected by the size of matrix will be obtained by dividing $-c_3$ by the quadratic (2).

As the result, we propose a new index of consistency:

$$c_3^* = \frac{-c_3}{n^2 - 7n + 13}. \tag{3}$$

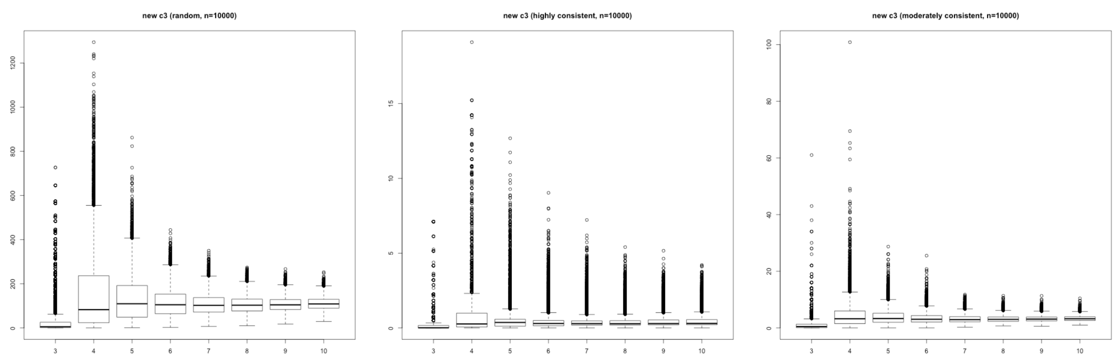

Figure 3: Box plots of c_3^* s for S_1, S_2 and S_3.

[4] We used the function for nonlinear least squares estimation, `nls()`, of statistical computing environment R.

[5] When we tried polynomial regression of other degree, such common polynomial was not observed.

Fig. 3 shows the distributions of c_3^* for S_1, S_2 and S_3. Although the results are not necessarily up to our expectations when $n = 3,4$, when $n \geq 5$, it is observed that c_3^* behaves as we prospected on the whole. Therefore, just for $n \geq 5$, c_3^* we proposed must be an index less affected by the size of matrix.

4. Reference value of consistency

Let's consider another purpose of this article. In order to see the correspondence between the values of CI and c_3^* around $\mathrm{CI} = 0.1$, we illustrate the scatter plots of CI vs c_3^* for S_3 (Fig. 4).

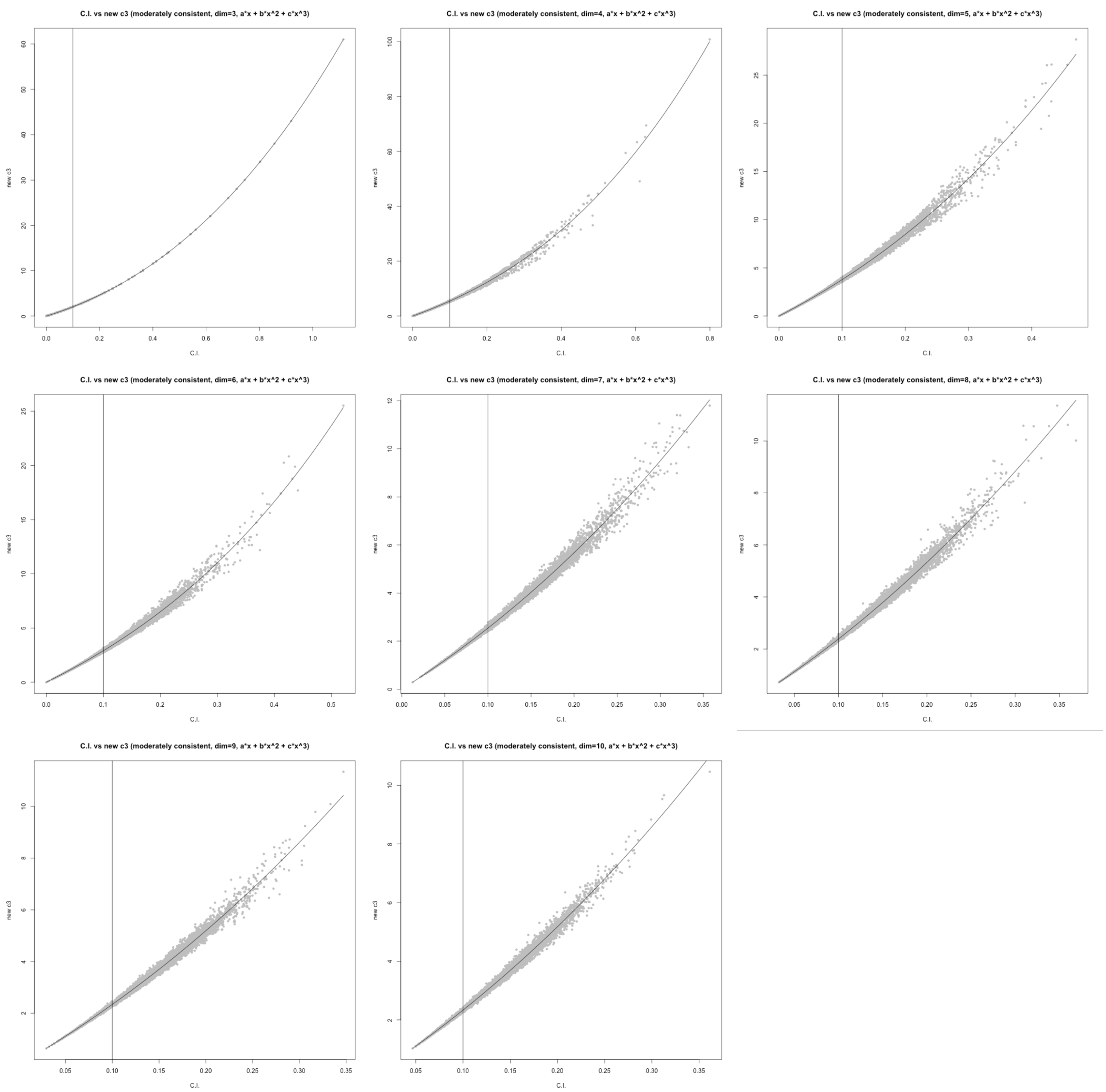

Figure 4: Scatter plots of CI vs c_3^* for S_3.

Table 1: Values of c_3^* corresponding to $\mathrm{CI} = 0.1$.

$n = 3$	2.048	$n = 7$	2.542
$n = 4$	5.473	$n = 8$	2.388
$n = 5$	3.783	$n = 9$	2.340
$n = 6$	2.902	$n = 10$	2.337

Fig. 4 also includes cubic polynomial regression curves. The reason why we use cubic polynomial is cubic correspondence (1) between CI and $-c_3$, i.e. c_3^*, when $n = 3$.

Table1 shows the predictive values of these regressions for $\mathrm{CI} = 0.1$. The values start 3.783 when $n \geq 5$ and seem to converge to about 2.3 as n increase. Then we propose $c_3^* \leq 2.5$ as a reference value of good consistency of pairwise comparison matrix.

5. Conclusion

In this paper, three methods for generating random pairwise comparison matrices were considered and the distributions of the values of indices Saaty's CI and $-c_3$ for matrices generated by these methods were examined. It was observed that the values of $-c_3$ increase quadratic polynomially as n increase. By dividing $-c_3$ by a quadratic of n, new index of consistency was proposed. It was observed that the proposed index is less affected by the size of matrix when $n \geq 5$. New reference value of good consistency of pairwise comparison matrix was proposed.

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